## Week 10:

**Recall:** Given a subspace $V \subseteq \mathbb{R}^m$. The dimension of $V$ is defined to be the number of element in a basis of $V$.

( * The number is independent of the choice of basis.)

In term of this terminology:

Thm (rephrase from discussion last week).

Suppose $V$ is a $k$-dim subspace in $\mathbb{R}^m$ (so that $k \leq m$), if $W$ is another subspace of $\mathbb{R}^m$ s.t. $W \subseteq V$, then $\dim(W) \leq \dim(V)$.

* Using the previous language, if $w_1, w_2, \cdots, w_n$ is a basis of $W$, then $n \leq k$, where $\begin{cases} n = \dim(W) \\ k = \dim(V). \end{cases}$

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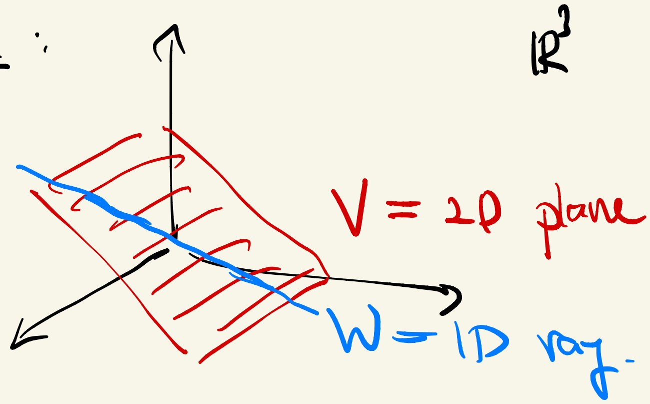
**Defn:** If $V, W$ are subspace in $\mathbb{R}^m$ s.t. $W \subseteq V$ then we say that $W$ is a subspace of $V$.

Eg picture: $\mathbb{R}^3$



Eg: $A = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 1 & 3 & 3 & 5 \\ 2 & 6 & 5 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 1 & 3 & 3 & 5 \end{bmatrix}$

then $\mathrm{Null}(A) = \{ x \in \mathbb{R}^4 \mid Ax = 0 \in \mathbb{R}^3 \}$

$\mathrm{Null}(B) = \{ x \in \mathbb{R}^4 \mid Bx = 0 \in \mathbb{R}^2 \}$

then $\mathrm{Null}(A)$ is a subspace of $\mathrm{Null}(B)$ since

① $\mathrm{Null}(A)$, $\mathrm{Null}(B)$ are both subspace of $\mathbb{R}^4$

② if $x \in \mathrm{Null}(A)$, then $Ax = 0 \Rightarrow Bx = 0$.

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Thm (Equality case): If $W$ is a subspace of $V \subseteq \mathbb{R}^n$, and $\dim(W) = \dim(V)$, then $W = V$.

pf: let $w_1, w_2, \cdots, w_n$ be a basis of $W$, $\dim W = \dim V = n$.

if $W \neq V$, then $\exists\, v_{n+1} \overset{\neq 0}{} \in V \setminus W$ s.t.

$$v_{n+1} \notin \mathrm{span}\{w_1, w_2, \cdots, w_n\} = W.$$

$\Rightarrow \{v_{n+1}, w_1, w_2, \cdots, w_n\}$ is linearly indep

$\Rightarrow \dim(V) \geq n+1$ $\rightarrow\leftarrow$.

$\therefore V = W$ #.

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Implication: If $v_1, v_2, \cdots, v_n \in V$ s.t. $n = \dim(V)$ and they are linearly indep. then they are automatically a basis.

Since $W = \text{span}\{v_1, v_2, \cdots, v_n\}$ has $\dim W = n = \dim V$

$\Rightarrow W = V$. #.

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rephrase some ideas in term of system of eqn:

Thm: let $V \subseteq \mathbb{R}^m$ be a subspace. Let $v_1, v_2, \cdots, v_p$ be vectors in $V$. Denote $A = [v_1 \; v_2 \; \cdots \; v_p]$, a $m \times p$ matrix.
Then the following are equivalent.

① $v_1, v_2, \cdots, v_p$ is a basis of $V$

* ② $\dim(V) = p$ and there are no non-trivial sol to $LS(A, 0)$.

* ③ $\dim(V) = p$ and " $LS(A, b)$ is consistent for all $b \in V$ ".

pf: ① $\Rightarrow$ ②

- $\dim(V) = p$ ✓
- If $Ax = 0$, write $x = \begin{bmatrix} x_1 \\ \vdots \\ x_p \end{bmatrix}$ s.t. $\sum_{i=1}^{p} x_i v_i = 0$

$\because$ $v_1, \cdots, v_p$ are linearly indep

$\therefore$ $x_i = 0 \quad \forall i = 1, 2, \cdots, p$. #.

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② $\Rightarrow$ ① : If $Ax = 0$ has no non-trivial sol

then $v_1, v_2, \cdots, v_p$ is linearly indep and hence is a basis.

① ⇒ ③ : trivial. Since $\forall b \in V_1$ $b = \sum_{i=1}^{p} \alpha_i v_i$ for some $\alpha_i \in \mathbb{R}$

as $V = \text{span}\{v_1, v_2, \cdots, v_p\}$.

③ ⇒ ① : Claim : $V = \text{span}\{v_1, v_2, \cdots, v_p\}$.

If Not, then $\exists\, w \overset{\neq 0}{} \in V \setminus \text{span}\{v_1, v_2, \cdots, v_p\}$.

Taking $b = w$, then $LS(A, b)$ admits no sol. $\rightarrow\leftarrow$

Then $v_1, v_2, \cdots, v_p$ is immediately linearly indep.

Since otherwise $\underset{\dim(V)}{\underset{\|}{\dim(\text{span}\{v_1, v_2, \cdots, v_p\})}} < p = \dim(V)$.

which is impossible!!

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Some important remark: (Math major only) Think of a "function" : $F: V \to V$ by $F(x) = Ax$.

2 : $Ax = 0$ has no non-trivial sol.

Implication : If $Ax = A\tilde{x} = y \in$ Image (one-to-one)

then $x = \tilde{x}$

3. $\forall y \in V$, $\exists x$ s.t. $Ax = y$.

Implication : If $y$ is in co-domain, then (onto)

$y$ is in range of $F$.

$\therefore$ FB 1-1 $\Leftrightarrow$ F is onto.

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## Some terminology :

Given a $m \times n$ matrix $A$.

- rank$(A)$ = Number of pivot column in RREF of $A$
- $C(A)$ = column space of $A$ $= \{y \in \mathbb{R}^m \mid y = Ax,\ x \in \mathbb{R}^n\}$
- $R(A)$ = row space of $A \triangleq C(A^t)$
- Null$(A)$ = null space of $A$

☆☆ $C(A)$, $R(A)$ are subspace of $\mathbb{R}^m$, $\mathbb{R}^n$ respectively.

$\rightsquigarrow$ $\dim(C(A))$ = column rank of $A$

$\rightsquigarrow$ $\dim(R(A))$ = row rank of $A$.

$\rightsquigarrow$ nullity$(A)$ $= \dim(\text{Null}(A))$.

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## Example ☆☆ :

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & 0 \\ 3 & 4 & 4 & 3 \\ 2 & 2 & 1 & 1 \end{bmatrix} \xrightarrow{\text{row op.}} A' = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$d_1 = 1$ ; $d_2 = 2$ ; $d_3 = 3$

$\therefore$ rank$(A) = 3 =$ rank$(A')$

- $C(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \\ 1 \end{bmatrix} \right\}$

  $= \text{span}\{u_1, u_2, u_3\}$ $\therefore \dim(C(A)) = 3$.

- $C(A') = \text{span}\{e_1, e_2, e_3\}$

- $\operatorname{Null}(A) = \{ x \in \mathbb{R}^n \mid Ax = 0 \} = \operatorname{Null}(A')$

$$\Rightarrow \operatorname{nullity}(A) = 1.$$

$$\operatorname{Nullity}(A) + \underset{\underset{\dim(C(A))}{\shortparallel}}{\operatorname{rank}(A)} = 4.$$

$$A^t = \begin{bmatrix} 1 & 1 & 3 & 2 \\ 1 & 0 & 4 & 2 \\ 1 & -1 & 4 & 1 \\ 1 & 0 & 3 & 1 \end{bmatrix} \longrightarrow (A^t)' = \begin{array}{c} \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{cccc} \uparrow & \uparrow & \uparrow & \phantom{-2} \\ d_1 & d_2 & d_3 & \phantom{-2} \end{array} \end{array}$$

$$\operatorname{rank}(A^t) = 3$$

$$\dim(R(A)) = \dim(C(A^t)) = 3 = \operatorname{rank}(A)$$

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Eg 2:

$$A = \begin{bmatrix} \overbrace{1 \quad 2 \quad 2 \quad 3 \quad 4}^{5} \\ 1 \quad 3 \quad 3 \quad 4 \quad 5 \\ 2 \quad 6 \quad 5 \quad 9 \quad 6 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 2 & -3 \\ 0 & 0 & 1 & -1 & 4 \end{bmatrix} = A'$$

$$A^t = B = \begin{bmatrix} \overbrace{1 \quad 1 \quad 2}^{3} \\ 2 \quad 3 \quad 6 \\ 2 \quad 3 \quad 5 \\ 3 \quad 4 \quad 9 \\ 4 \quad 5 \quad 6 \end{bmatrix} \longrightarrow B' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- $\operatorname{rank}(A) = 3 = \operatorname{rank}(B)$.
- $\dim(C(A)) = 3$
- $\dim(R(A)) = \dim(C(A^t)) = 3$
- $\operatorname{nullity}(A) = 2$
- $\operatorname{nullity}(B) = 0$

$$\operatorname{rank}(A) + \operatorname{nullity}(A) = 3 + 2 = 5$$

$$\operatorname{rank}(B) + \operatorname{nullity}(B) = 3 + 0 = 3$$

In general:

Rank-nullity thm: Suppose $A$ is a $m \times n$ matrix.

then ① $\text{rank}(A) = \dim(C(A)) = \dim(R(A))$

② $\text{rank}(A) + \text{nullity}(A) = n$.

③ $\text{rank}(A^t) = \text{rank}(A)$.

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Pf: ①: $\text{rank}(A) =$ number of pivot column in $A'$ (= RREF of $A$)

$= \dim(C(A'))$ $(= \text{rank}(A'))$

$= \dim(C(A))$

①: $\dim(R(A)) =$ no. of element of non-zero row in $A'$

$=$ No. of pivot column in $A'$

$= \text{rank}(A)$.

②: $\text{Null}(A) = \text{Null}(A')$

$\dim(\text{Null}(A')) =$ no. of free column in $A'$

$= n -$ no. of pivot column of $A'$

$= n - \text{rank}(A')$

$\therefore$ $\text{Nullity}(A) = n - \text{rank}(A)$.

③: $\operatorname{rank}(A^t) = \dim(R(A^t)) = \dim(C(A)) = \operatorname{rank}(A)$. #

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Some conseq.: (no need to memorize, but understand the importance of rk thm)

Thm: Given a $m \times n$ matrix $A$.

(a) $\operatorname{rank}(A) \leq \min\{m, n\}$

(b) $\operatorname{nullity}(A) \geq n-m$

pf: $\operatorname{rank}(A) = \dim(C(A)) \leq n$ $\Rightarrow$ (a).

$= \dim(R(A)) \leq m$.

$\operatorname{nullity}(A) = n - \operatorname{rank}(A) \geq n-m$. #

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Thm: Suppose $A = p \times q$ matrix

$B = q \times s$ matrix, then

(a) $\operatorname{nullity}(B) \leq \operatorname{nullity}(AB)$

(b) $\operatorname{rank}(AB) \leq \operatorname{rank}(B)$

(c) $\operatorname{rank}(AB) \leq \operatorname{rank}(A)$

(d) $\operatorname{nullity}(A) + s \leq \operatorname{nullity}(AB) + q$.

pf: (a) $\operatorname{Null}(B) = \{x \in \mathbb{R}^s \mid Bx = 0\}$

$\subseteq \{x \in \mathbb{R}^s \mid ABx = 0\} = \operatorname{Null}(AB)$

$\Rightarrow$ $\dim(\mathrm{Null}(B)) \leq \dim(\mathrm{Null}(AB))$ (by ineq between subspaces).

(a). $C(AB) = \{y \in \mathbb{R}^p \mid y = ABx \text{ for some } x \in \mathbb{R}^s\}$

$\subseteq \{y \in \mathbb{R}^p \mid y = Az \text{ for some } z \in \mathbb{R}^q\}$

$= C(A)$

$\therefore$ $\mathrm{rank}(AB) \leq \mathrm{rank}(A)$.

(b). $\mathrm{rank}(AB) = \mathrm{rank}(B^t A^t)$

$\leq \mathrm{rank}(B^t) = \mathrm{rank}(B)$.

(d). $\begin{cases} \mathrm{nullity}(A) + \mathrm{rank}(A) = q \\ \mathrm{nullity}(AB) + \mathrm{rank}(AB) = s \end{cases}$

$\Rightarrow$ $\mathrm{nullity}(A) - q \leq \mathrm{nullity}(AB) - s$ by (c).

$\Rightarrow$ (d) #

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new topic: Eigenvalues and Eigenvectors.

Given a square matrix $A$ ($n \times n$ matrix),

work: "simplify" $A$ ??

Defn: A vector $\vec{v} \neq 0 \in \mathbb{R}^n$ is said to be a eigenvector of $A$ with eigenvalue $\lambda$ if $Av = \lambda v$.

observation: ① If $Av = \lambda v$, then $A(\beta v) = \beta Av = \lambda(\beta v)$. for $\beta \in \mathbb{R}$. ($\beta v$ is also eigvector)

② If $v_1, v_2$ are eignvector of $A$ with eignvalue $\lambda$, then

$$A(v_1 + v_2) = Av_1 + Av_2 = \lambda(v_1 + v_2).$$

($v_1 + v_2$ is also eignvector with eignvalue $\lambda \in \mathbb{R}$)

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Eg: ⓐ $A = \begin{bmatrix} 13 & 30 \\ -6 & -14 \end{bmatrix}$, $u = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$, $v = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

- $Au = \begin{bmatrix} 5 \\ -2 \end{bmatrix} = u$. (eignvalue $= 1$.)

- $Av = \begin{bmatrix} -4 \\ 2 \end{bmatrix} = -2v$ (eignvalue $= -2$)

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ⓑ $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$, $u = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $v = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $w = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

then $\cdot Au = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix} = 4u$

- $Av = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = v$ ⟵ same eigenvalues, but eigenvectors not parallel.

- $Aw = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = w$

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(c) $A = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$ then $\begin{cases} Ae_1 = \lambda_1 e_1 \\ Ae_2 = \lambda_2 e_2 \\ Ae_3 = \lambda_3 e_3 \end{cases}$

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Q: Is it always possible to find eigenvectors (and eigenvalues)??

non-example:

① $A = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$ :

goal: find eigenvalue of $A$ if exists.

Suppose $v$ satisfies $Av = \beta v$.

$\Rightarrow (A - \beta I)v = 0 \Rightarrow v \in \mathrm{Null}(A - \beta I)$.

$A - \beta I = \begin{bmatrix} \lambda-\beta & 1 & 0 \\ 0 & \lambda-\beta & 1 \\ 0 & 0 & \lambda-\beta \end{bmatrix}$ is non-singular iff $\lambda - \beta \neq 0$.

$\Rightarrow \beta$ must be $\lambda$. $\Rightarrow v \in \mathrm{Null}(A - \lambda I)$

$$A-\lambda I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$v \in \mathrm{Null}(A-\lambda I) = \mathrm{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}$$

$\therefore$ eigenvector $= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ with eigenvalue $= \lambda$.

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② $A = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$.

If $v$ is a eigenvector with eigenvalue $\lambda$,

$\Rightarrow$ $A-\lambda I$ must be singular where

$$A-\lambda I = \begin{bmatrix} 1-\lambda & 0 & 0 & -1 \\ 1 & 1-\lambda & 0 & 0 \\ 0 & 1 & 1-\lambda & 0 \\ 0 & 0 & 1 & 1-\lambda \end{bmatrix}$$

$$\Rightarrow \begin{cases} v_4 = (1-\lambda) v_1 \\ v_1 = -(1-\lambda) v_2 \\ v_2 = -(1-\lambda) v_3 \\ v_3 = -(1-\lambda) v_4 \end{cases} \Rightarrow v_4 + (1-\lambda)^4 v_4 = 0$$

$\Rightarrow v_4 = 0$

$\Rightarrow v_1 = v_2 = v_3 = v_4 = 0$

which is impossible.

$\therefore$ A has no eigenvalues.

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Q: If $\begin{cases} Av = \lambda v \\ Au = \tilde{\lambda} u \end{cases}$ with $\lambda \neq \tilde{\lambda}$, how $v$, $u$ are related??

<u>Lemma</u>: They are linearly indep.

pf: If $\alpha u + \beta v = 0$, then $\alpha \lambda u + \beta \tilde{\lambda} v = 0$.

hence, $\alpha \lambda u + \beta \lambda v = 0 = \alpha \lambda u + \beta \tilde{\lambda} v$.

$$\Rightarrow \beta(\lambda - \tilde{\lambda}) v = 0$$

$$\Rightarrow \beta = 0 \Rightarrow \alpha = 0 \ \#.$$

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